APPLYING MULTIDIMENSIONAL SPACE GEOMETRICAL REPRESENTATION METHODS IN THE GRAPHICAL EXPRESSION OF THE TRIPLE INTEGRAL

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Abstract: Generally, the shapes of surfaces can be defined by equations in an analytical way (cylinders, spheres, hyperboloids, etc.). In most cases, such an analytical approach can not be taken and the designer must create surfaces form simple elements, such as curves or points. The problem we approach comes from the needs of the industry (automotive, aeronautics, naval, etc.). When a prototype is conceived, a scale model needs to be fabricated in order to verify if the stylists' designs can be realized. This shows a need to have the equations of different surfaces that define the object or, more accurately a good approximation of these surfaces.

Keywords: descriptive geometry, triple integral, axonometric drawing.

1. INTRODUCTION

The qualities expected form different mathematical models are: supporting (insuring) good physical elements through a good behavior of primary derivates (adjustment points for the checkered surface) and secondary derivates (the curvature of surfaces); the facility to modify the shape of surfaces starting from parameters with a physical meaning that is easy to understand and manipulate by non-mathematicians; the calculus facility, these methods being designed for an interactive use.

2. A GEOMETRICAL APPROACH TO THE TRIPLE INTEGRAL

We know that a double integral such as:

$$\iint_{\sigma} f(x, y)\, dx\, dy \tag{1}$$

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is geometrically expressed by the volume V of the shape, limited by a part of the surface S, whose equation

$$z = f(x, y) \tag{2}$$

is the projection σ of this part on the *xOy* coordinate plain and the cylindrical surface projected with generators, parallel to the *Oz* axis, of the *Oxyz* orthogonal coordinate system, to which all analyzed objects are reported.

A physical approach to the double integral takes us to the notion of mass, distributed on the plain domain σ according to the law that is expressed by function (2). Analyzing the analogical mass distributed on the space domain V, in such a way, that its distribution law is expressed with the help of the function:

$$t = f(x, y, z) \tag{3}$$

we normally get to the physical approach to the triple integral of the following type:

$$\iiint_V f(x, y, z)\, dx\, dy\, dz \tag{4}$$

A geometrical approach to the type (4) integral can not be found in integral calculus books, and the cause is obviously the fact that function (3) itself includes four variables and in order to express its geometrical content we need to operate with the notions of the four-dimensional geometrical space, that are not widely spread yet.

The function $t = f(x, y, z)$ is expressed through a certain surface of the four-dimensional space related to the orthogonal coordinate system *Oxyzt*. So, by expanding the geometrical approach for the double integral to the triple integral, we can conclude that the triple integral of type (4) is geometrically expressed by volume H of the hyper-corps, limited by a part Ω of hyper-surface (3), by projecting this part on the hyper-plain of *xyz* coordinates and through the cylindrical surface projected with the generators parallel to the *Ot* axis.

There are known examples of representing on orthogonal and axonometric drawings the construction of hyper-surfaces.

Based upon these facts we can conclude that using the methods of descriptive geometry in the four-dimensional space, we can eloquently express the geometrical substance of triple integration.

3. THE GRAPHICAL EXPRESSION OF TRIPLE INTEGRATION IN THE AXONOMETRIC DRAWING

With the help of the calculus example for the mass of the corpus limited by the surface of the ellipsoid:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \tag{5}$$

we show how the triple integration process can be suggestively represented in the axonometric drawing.

We presume that the density in each point of the corpus is equal to the square of the distance from this point to the starting point of the coordinates. In such a case, the mass distribution law, according to the volume of the given corpus, will be expressed by the function.

$$t = x^2 + y^2 + z^2 \tag{6}$$

And the mass m of the given corpus by the integral

$$m = \iiint\limits_V \left(x^2 + y^2 + z^2\right) dx\, dy\, dz \tag{7}$$

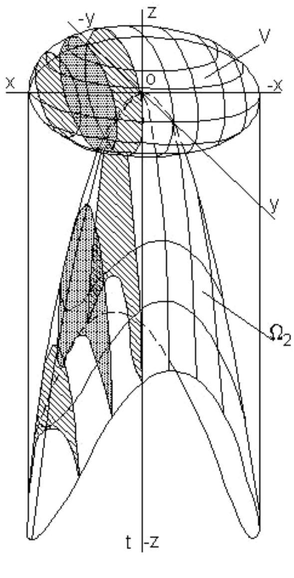

Fig. 1. Representing the geometrical object that expresses the integral in diametric oblique projection

Figure 1, in the dimetric oblique projection shows the geometrical object that expresses integral (7) as a whole. On this drawing V is the volume of the ellipsoid (5) which is the orthogonal projection of the hyper-function (6) on the hyper-plain of *xyz* coordinates, Ω_2 is the projection of the hyper-surface (6) on the hyper-plain of *xyz* coordinates on a direction parallel to the *zt* coordinate plain, chosen in such a way that the positive direction of the *Ot* axis is projected on the negative direction of the *Oz* axis.

Integral (7) can be represented as follows:

$$\int_{z_1}^{z_2} dz \int_{x_1(z)}^{x_2(z)} dx \int_{y_1(x,z)}^{y_2(x,z)} f(x,y,z)\,dy \tag{8}$$

or, having in mind the actual values of the integration limits in function (3), as represented below

$$\int_{-c}^{c} dz \int_{-\frac{a}{c}\sqrt{c^2-z^2}}^{\frac{a}{c}\sqrt{c^2-z^2}} dx \int_{-b\sqrt{1-\frac{x^2}{a^2}-\frac{z^2}{c^2}}}^{b\sqrt{1-\frac{x^2}{a^2}-\frac{z^2}{c^2}}} \left(x^2+y^2+z^2\right) dy \tag{9}$$

If integrating after y the variables x and z remain constant and the interior integral takes the following values

$$\int_{-b\sqrt{1-\frac{x^2}{a^2}-\frac{z^2}{c^2}}}^{b\sqrt{1-\frac{x^2}{a^2}-\frac{z^2}{c^2}}} \left(x^2+y^2+z^2\right) dy = 2\left(x^2+z^2\right) b\sqrt{1-\frac{x^2}{a^2}-\frac{z^2}{c^2}} + \frac{2}{3} b^3 \left(\sqrt{1-\frac{x^2}{a^2}-\frac{z^2}{c^2}}\right)^3 \tag{10}$$

The geometrical essence of such an integral is expressed by the surface represented in figure 2.a.

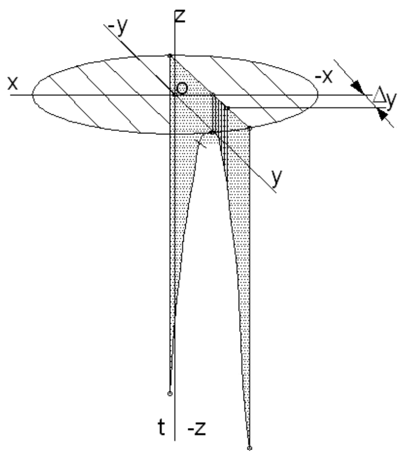

Fig. 2.a. Representing the surface

Integral (10) is described as follows:

$$2\int_{-c}^{c} dz \int_{-\frac{a}{c}\sqrt{c^2-z^2}}^{\frac{a}{c}\sqrt{c^2-z^2}} \left[\, x^2\, b \sqrt{1-\frac{x^2}{a^2}-\frac{z^2}{c^2}} + z^2\, b \sqrt{1-\frac{x^2}{a^2}-\frac{z^2}{c^2}} + \right.$$
$$\left. + \frac{b^3}{3}\sqrt{1-\frac{x^2}{a^2}-\frac{z^2}{c^2}}\left(1-\frac{x^2}{a^2}-\frac{z^2}{c^2}\right)\,\right] dx \qquad (11)$$

If we note $(a/c)\sqrt{c^2-z^2} = d$ we get:

$$2\int_{-c}^{c} dz \int_{-d}^{d} \left[\frac{b}{a}\, x^2 \sqrt{d^2-x^2} + \frac{b^3}{3\,a^3}\sqrt{d^2-x^2}\left(d^2-x^2\right) + \frac{b\,c^2}{a}\left(1-\frac{d^2}{a^2}\right)\sqrt{d^2-x^2}\right] dx \qquad (12)$$

If integrating after x variable z remains constant and the interior integral takes the following values:

$$2\int_{-d}^{d} \left[\frac{b}{a}\, x^2 \sqrt{d^2-x^2} + \frac{b^3}{3\,a^3}\sqrt{d^2-x^2}\left(d^2-x^2\right) + \frac{b\,c^2}{a}\left(1-\frac{d^2}{a^2}\right)\sqrt{d^2-c^2}\right] dx =$$
$$= \frac{\pi}{4}\frac{a\,b}{c^4}\left(a^2+b^2-4\,c^2\right)\left(c^4-2\,c^2\,z^2+z^4\right) \qquad (13)$$

From a geometrical point of view the result of such integration is expressed by the volume represented in figure 2.b, where the element of this volume is reduced (brought) to the surface represented earlier in figure 2.a. In the final stage of integration we substitute the value of the internal integral in expression (12) and we integrate after z.

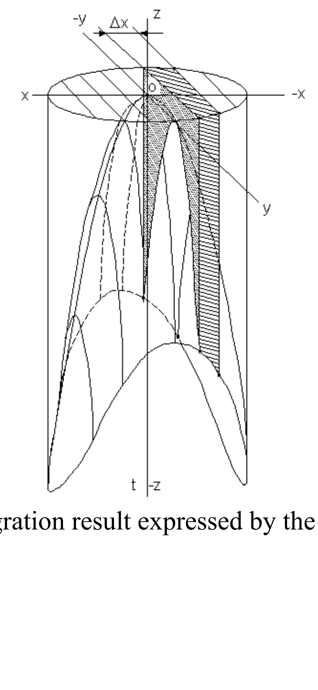

Fig. 2.b. The integration result expressed by the represented volume

$$\frac{\pi}{4}\frac{a\,b}{c^4}\left(a^2+b^2+4\,c^2\right)\cdot\int_{-c}^{c}\left(c^4-2\,c^2\,z^2+z^4\right)dz=\frac{4\,\pi}{15}\,a\,b\,c\left(a^2+b^2+c^2\right) \qquad (14)$$

The process of such integration is geometrically expressed by the summing of volumes, out of which one is indicated in figure 2.b. Following this summing, a hyper-volume is formed, whose element is shown in figure 3. This drawing also shows the elements presented before in figure 2.

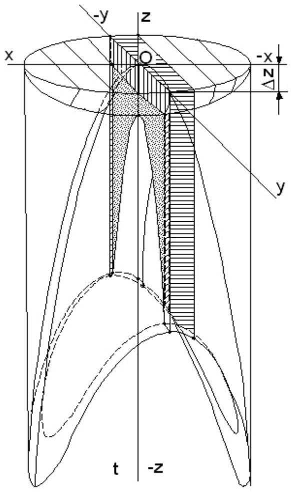

Fig. 3. Representing the hypervolume

The expression $\frac{4}{15}\pi abc(a^2+b^2+c^2)$ is the value of the triple integral (7).

2. CONCLUSION

This example shows that, by using four-dimensional space descriptive geometry methods, we can eloquently express the essence of the triple integral.

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